Concepts of Mathematics

October 22, 2012

Lecture 12-13

1 Principles of Mathematical Induction

Consider the following three statements, each involving a general positive integer n.

(1) The sum of the first n odd numbers is equal to n^2.

(2) If $p \geq -1$ then $(1+p)^n \geq 1+np$.

(3) The sum of the internal angles of an n-sided convex polygon is $(n-2)\pi$.

Proposition 1 (Mathematical Induction)**.** *Suppose for each positive integer n we have a statement $P(n)$. If we prove the following two things:*

(a) (Induction Basis) $P(1)$ *is true;*

(b) (Induction Hypothesis) *If* $P(n)$ *is true then* $P(n+1)$ *is also true;*

Then $P(n)$ *is true for all positive integers* n.

The logic is clear: $P(1) \Rightarrow P(2)$, $P(2) \Rightarrow P(3)$, $P(2) \Rightarrow P(3)$, ... It follows that $P(n)$ is true for all positive integers n.

Example 1. Let $P(n)$ denote the statement: $1+3+5+\cdots+(2n-1) = n^2$, where $n \geq 1$.

Proof. (a) $P(1)$ is true: $1 = 1^2$.

(b) Suppose $P(n)$ is true. Then $1+3+5+\cdots+(2n-1) = n^2$. Adding $2n+1$ to both sides we have

$$1+3+5+\cdots+(2n-1)+(2n+1) = n^2+2n+1 = (n+1)^2.$$

This means that the statement $P(n+1)$ is true. Thus $P(n)$ is true for all positive integers n. □

Example 2. Let $P(n)$ denote the statement: If $p \geq -1$ then $(1+p)^n \geq 1+np$ for all positive integers n.

Proof. (a) $P(1)$ is true: $1+p \geq 1+p$.

(b) Suppose $P(n)$ is true. Then $(1+p)^n \geq 1+np$. Since $p \geq -1$, then $p+1 \geq 0$. Multiplying $1+p$ to both sides, we have

$$(1+p)(1+p)^n \geq (1+p)(1+np) = 1+np+p+np^2 \geq 1+(n+1)p.$$

This is exactly the statement $P(n+1)$. Thus $P(n)$ is true for all positive integers n. □

Proposition 2 (Mathematical Induction)**.** *Let k be an integer. Suppose for each integer $n \geq k$ we have a statement $P(n)$. If we prove the following two things:*

(a) (Induction Basis) *$P(k)$ is true;*

(b) (Induction Hypothesis) *If $P(n)$ is true then $P(n+1)$ is also true;*

Then $P(n)$ is true for all integers $n \geq k$.

Example 3. Let $P(n)$ denote the statement: The sum of internal angles of an n-sided convex polygon is $(n-2)\pi$.

Proof. Note that n must be an integer larger than or equal to 3, i.e., $n \geq 3$.

(a) $P(3)$ is true: *checked in junior high school.*

(b) Suppose $P(n)$ is true. Let $A_1A_2\cdots A_{n+1}$ be an $(n+1)$-sided polygon whose vertices are $A_1, A_2, \ldots, A_{n+1}$. Draw a segment between the two vertices A_1 and A_{n+1}. We have a triangle $\Delta A_1A_nA_{n+1}$ and an n-sided polygon $A_1A_2\ldots A_n$. Let α_i denote the internal angle of the polygon $A_1A_2\cdots A_{n+1}$ at the vertex A_i, β_j the internal angle of the polygon $A_1A_2\cdots A_n$ at the vertex A_j, and γ_k the internal angle of $\Delta A_1A_nA_{n+1}$ at the vertex A_k. Then $\alpha_1 = \beta_1 + \gamma_1$, $\alpha_n = \beta_n + \gamma_n$, $\alpha_{n+1} = \gamma_{n+1}$, and $\alpha_i = \beta_i$ $(2 \leq i \leq n-1)$. Thus

$$\begin{aligned}
\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} &= (\beta_1 + \gamma_1) + \beta_2 + \cdots + \beta_{n-1} + (\beta_n + \gamma_n) + \gamma_{n+1} \\
&= (\gamma_1 + \gamma_n + \gamma_{n+1}) + (\beta_1 + \cdots + \beta_{n-1} + \beta_n) \\
&= \pi + (n-2)\pi = (n-1)\pi = \big((n+1)-2\big)\pi.
\end{aligned}$$

This means that the statement $P(n+1)$ is true. So $P(n)$ is true for all integers $n \geq 3$. □

Example 4. Let $P(n)$ denote the statement: $2^n < n!$, where $n \geq 1$.

Proof. (a) $P(1)$ is true: Trivial.

(b) Suppose $P(n)$ is true, i.e., $2^n < n!$. Then

$$2^{n+1} = 2 \cdot 2^n < (n+1) \cdot n! = (n+1)!.$$

This is exactly the statement $P(n+1)$. Thus $P(n)$ is true for all positive integers n. What is wrong? □

Consider the problem: $2^n \leq n!$ for $n \geq 0$.

Proposition 3 (Strong Mathematical Induction)**.** *Let k be an integer. Suppose for each integer $n \geq k$ we have a statement $P(n)$. If we prove the following two things:*

(a) (Induction Basis) *$P(k)$ is true.*

(b) (Strong Induction Hypothesis) *If $P(k), P(k+1), \ldots, P(n)$ are true then $P(n+1)$ is also true.*

Then $P(n)$ is true for all integers $n \geq k$.

Example 5. Let $P(n)$ denote the statement: $u_n = 2^n + 1$, where u_n is the sequence with $n \geq 0$, $u_0 = 2$, $u_1 = 3$, and

$$u_{n+1} = 3u_n - 2u_{n-1}, \quad n \geq 1.$$

Proof. (a) $P(1)$ is true: $2 = 1 + 1 = 2^0 + 1$.

(b) Suppose $P(n)$ is true, i.e., $u_n = 2^n + 1$. Then by induction hypothesis

$$u_{n+1} = 3u_n - 2u_{n-1} = 3 \cdot (2^n + 1) - 2 \cdot (2^{n-1} + 1) = 2^{n+1} + 1.$$

This means that the statement $P(n+1)$ is true. Thus $P(n)$ is true for all integers $n \geq 0$. □

Example 6. Find a closed formula for the sum $1^2 + 2^2 + \cdots + n^2$.

Proof. It is known that $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$. It suggests that the wanted closed formula for the sum is a polynomial of degree 3. Set

$$1^2 + 2^2 + \cdots + n^2 = a_0 + a_1 n + a_2 n^2 + a_3 n^3.$$

Then for $n = 1, 2, 3, 4$, we have the system of linear equations

$$\begin{cases} a_0 & +a_1 & +a_2 & +a_3 & = & 1 \\ a_0 & +2a_1 & +2^2 a_2 & +2^3 a_3 & = & 5 \\ a_0 & +3a_1 & +3^2 a_2 & +3^3 a_3 & = & 14 \\ a_0 & +4a_1 & +4^2 a_2 & +4^3 a_3 & = & 30 \end{cases}$$

By the Cramer's rule we have that the unique solution: $a_0 = 0$, $a_1 = \frac{1}{6}$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$. Now we prove by mathematical induction the proposed formula

$$P(n): \quad 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

(a) $P(1)$ is true: $1 = \frac{1}{6} \cdot 2 \cdot 3$.

(b) Suppose $P(n)$ is true. Then by induction hypothesis

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \\ &= \frac{1}{6}(n+1)\big[n(2n+1) + 6(n+1)\big] \\ &= \frac{1}{6}(n+1)(2n^2 + 7n + 6) \\ &= \frac{1}{6}(n+1)(n+2)(2n+3) \\ &= \frac{1}{6}(n+1)\big[(n+1) + 2\big]\big[2(n+1) + 3\big]. \end{aligned}$$

This means that the statement $P(n+1)$ is true. Thus $P(n)$ is true for all integers $n \geq 1$. □

Definition 4. A **prime number** is a positive integer p such that $p \geq 2$ and the only positive integers dividing p are 1 and p.

Proposition 5. *Every positive integer greater than* 1 *is equal to a product of prime numbers.*

Proof. For each positive integer $n \geq 2$, let $P(n)$ denote the statement: the integer n is equal to a product of prime numbers.

(a) $P(2)$ is true, since 2 is a prime number.

(b) $P(2), P(3), \ldots, P(n) \Rightarrow P(n+1)$: For the positive integer $n+1$, if $n+1$ is a prime p, then $n + 1 = p$ is already a product of prime numbers (only one prime number in the product); if $n+1$

is not a prime number, then there is a positive integer a dividing $n+1$. Writing $b = \frac{n+1}{a}$, we have b is an integer, $b \geq 2$, and

$$n+1 = ab, \quad \text{where } a, b \in \{2, 3, \ldots, n\}.$$

By induction hypothesis, the positive integers a and b have prime factorizations, say, $a = p_1 p_2 \cdots p_k$ and $b = q_1 q_2 \cdots q_l$. Then $n+1 = p_1 p_2 \cdots p_k q_1 q_2 \cdots q_l$. This means that $n+1$ is a product of prime numbers. □

2 Euler's Formula and Platonic Solids

A **polyhedron** is a solid whose surface consists of a number of **faces**, all of which are convex polygons, such that any side lies on exactly one other face. The corners of the faces are called **vertices** of the polyhedron, and their sides are the **edges** of the polyhedron.

A polyhedron is said to be **convex** if, whenever we choose two points on its surface, the straight line joining them lies entirely within the polyhedron.

Example 7. (a) A cube has 8 vertices, 12 edges, and 6 faces.

(b) A tetrahedron has 4 vertices, 6 edges, and 4 faces.

(c) The prism whose base is a rectangle is a polyhedron, having 5 vertices, 8 edges, and 5 faces.

Theorem 6 (Euler's Formula)**.** *For any convex polyhedron with V vertices, E edges, and F faces, we have the relation*

$$V - E + F = 2.$$

Proof. It follows from the following theorem. □

Definition 7. A **graph** is a figure in the plane consisting of a collection of points (called **vertices**) and some edges joining various pairs of these points. A graph is **connected** if we can get from any vertex of the graph to any other vertex by going along a path of edges in the graph. A graph is called a **plane graph** if there are no two edges crossing each other. A **loop** in a graph is an edge that joins two identical vertices.

A plane graph separate the plane into some connected regions.

Theorem 8 (Euler's Relation)**.** *For any connected plane graph G with v vertices, e edges, and r regions, we have*

$$v - e + r = 2.$$

Proof. Let $P(n)$ be the statement: every connected plane graph with n edges satisfies the formula $v - e + r = 2$. Note that $P(n)$ is a statement about lots of plane graphs. For instance, $P(1)$ is a statement about two plane graphs: a segment graph and a loop graph. We apply mathematical induction to prove $P(n)$ for all positive integers n.

(a) $P(1)$ is true: A segment graph has 2 vertices, 1 edge, and 1 face; so $2 - 1 + 1 = 2$. A loop graph has 1 vertex, 1 edge, and 2 faces; so $1 - 1 + 2 = 2$.

(b) Suppose $P(n)$ is true for n, i.e., every plane graph with n edges satisfies the Euler formula. Let G be a plane graph with $n+1$ edges. We have two cases.

CASE 1: *G has a bounded region.* Let x be an edge of G bounding a bounded region; and let G' be a graph obtained from G by removing the edge x. It is clear that G' is connected and planar, $v(G) = v(G')$, $e(G) = e(G') + 1$, and $r(G) = r(G') + 1$. Since $v(G') - e(G') + r(G') = 2$, we have

$$v(G) - e(G) + r(G) = v(G') - \big[e(G') + 1\big] + \big[r(G') + 1\big] = v(G') - e(G') + r(G') = 2.$$

This means that $P(n+1)$ is true.

CASE 2: *G has no bounded region.* Then G has the only unbounded region. So G has no closed path. It follows that G has an end-vertex, a vertex joined by only one edge. [Otherwise, if each vertex is joined by two edges, then we can start to travel on edges to obtain a closed path from one vertex, reaching an vertex through an edge and leaving the same vertex through another edge.] Take a end-vertex of G and remove the end-vertex and the only edge joining to it; we obtain a connected plane graph G'. Note that $v(G) = v(G') + 1$, $e(G) = e(G') + 1$, and $r(G) = r(G')$. then

$$v(G) - e(G) + r(G) = \big[v(G') + 1\big] - \big[e(G') + 1\big] + r(G') = v(G') - e(G') + r(G') = 2.$$

Now we have seen that $P(n+1)$ is true. The proof is finished. □

3 Regular and Platonic Solids

A convex polygon is said to be **regular** if all its sides are of equal length and all its internal angles are equal. A polyhedron is said to be **regular** if (i) all its faces (convex polygons) are regular and have the same number of sides; (ii) all vertices have the same number of edges joining them. The Platonic solids are the five regular polyhedra: *cube, tetrahedron, octahedron, dodecahedron, and icosahedron.*

Theorem 9. *The only regular convex polyhedra are the five Platonic solids.*

Proof. Let P be a regular polyhedron with v vertices, e edges, and f faces. Let n be the number of sides of a face, and d the number of edges joining a vertex. Then

$$2e = nf,$$

[It follows from the counting of the number of ordered pairs (ε, σ), where ε is an edge, σ is a face, and ε bounds σ.]

$$2e = dv.$$

[It follows from the counting of the number of ordered pairs (ν, ε), where ν is a vertex, ε is an edge, and ε joins ν.] Thus

$$f = \frac{2e}{n}, \quad v = \frac{2e}{d}.$$

Recall the Euler formula $v - e + f = 2$; we have $\frac{2e}{d} - e + \frac{2e}{n} = 2$. Dividing both sides by $2e$, we have

$$\frac{1}{d} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2}. \tag{1}$$

Note that $n \geq 3$, as a convex polygon must have at least 3 sides; likewise $d \geq 3$, since it is geometrically clear that in a polyhedron a vertex must belong to at least 3 edges. Since the right hand side of (1) is at least $\frac{1}{2}$, it follows that we cannot have both $d \geq 4$ and $n \geq 4$. So we have either $d \leq 3$ or $n \leq 3$, and subsequently either $d = 3$ or $n = 3$.

CASE $d = 3$. Then (1) becomes

$$\frac{1}{n} = \frac{1}{e} + \frac{1}{6}.$$

Since e is positive, it follows that $3 \leq n \leq 5$. So $(n, e) = (3, 6)$, $(4, 12)$, $(5, 30)$; i.e., $(v, e, f) = (4, 6, 4), (8, 12, 6), (20, 30, 12)$.

CASE $n = 3$. Then (1) becomes

$$\frac{1}{d} = \frac{1}{e} + \frac{1}{6}.$$

Since e is positive, it follows that $3 \leq d \leq 5$. So $(d, e) = (3, 6)$, $(4, 12)$, $(5, 30)$; i.e., $(v, e, f) = (4, 6, 4), (6, 12, 8), (12, 30, 20)$.

We thus have five regular polyhedra: tetrahedron $(4, 6, 4)$; cube $(8, 12, 6)$; octahedron $(6, 12, 8)$; dodecahedron $(20, 30, 12)$; icosahedron $(12, 30, 20)$. $\square$

A **complete graph** K_n is a graph with n vertices such that every two vertices are adjacent by an edge. The complete graph K_5 is not planar. Since $v(K_5) = 5$, $e(K_5) = 10$, if K_5 is planar then by the Euler formula we have $f(K_5) = 2 - v + e = 7$, i.e., K_5 has 7 faces. Since $2e \geq 3f$, it follows that $20 = 2e \geq 3f = 21$, this is a contradiction.

A complete bipartite graph is a graph $K_{m,n}$ whose vertex set can be divided into two parts V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$, and the edges set is $V_1 \times V_2$. The complete bipartite graph $K_{3,3}$ is non-planar. Note that $v = 6$ and $e = 9$. If $K_{3,3}$ is planar, then by the Euler formula we have $r = 2 - v + e = 5$ regions. Note that every cycle of a bipartite graph has even length, so every cycle of $K_{3,3}$ has length at least 4. Thus $2e \geq 4f$ implies $18 \geq 20$. This is a contradiction.

Example 8. The football graph has faces of pentagons and hexagons. Every vertex shares 3 edges and ever edge shares 2 vertices. Each pentagon is surrounded by 5 hexagons and each hexagon is surrounded by 3 pentagons. Find the number of vertices, edges, pentagons, and hexagons of the football graph.

Let v, e be the number of vertices and edges respectively. Let f_5, f_6 be the number of pentagons and hexagons. Then

$$3v = 2e, \quad 5f_5 + 6f_6 = 2e, \quad v - e + f_5 + f_6 = 2.$$

Note that the number of edges shared by both pentagons and hexagons is counted in two ways: counting by pentagons, counting by hexagons. We then have $5f_5 = 3f_6$.

Put $e = 3v/2$ into other equations, we have

$$v - \frac{3}{2}v + f_5 + f_6 = 2, \quad 5f_5 + 6f_6 = 3v, \quad 5f_5 = 3f_6.$$

Thus $f_5 = \frac{3}{5}f_6$.

$$-\frac{1}{2}v + \frac{3}{5}f_6 + f_6 = 2, \quad 3f_6 + 6f_6 = 3v.$$

$$v = 3f_6, \quad -5v + 16f_6 = 20.$$

$$f_6 = 20, \quad v = 60, \quad e = 90, \quad f_5 = 12.$$